The Zariski-Lipman conjecture for complete intersections

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Abstract

The tangential branch locus $B^t_{X/Y} \subset B_{X/Y}$ is the subset of points in the branch locus where the sheaf of relative vector fields $T_{X/Y}$ fails to be locally free. It was conjectured by Zariski and Lipman that if V/k is a variety over a field k of characteristic 0 and $B^t_{V/k} = \emptyset$, then V/k is smooth (=regular). We prove this conjecture when V/k is a locally complete intersection. We prove also that $B^t_{V/k} = \emptyset$ implies $\operatorname{codim}_X B_{V/k} \leq 1$ in positive characteristic, if V/k is the fibre of a flat morphism satisfying generic smoothness.

1 Introduction

Let $\pi : X \to Y$ be a morphism of nœtherian schemes which is locally of finite type, $\Omega_{X/Y}$ its sheaf of Kähler differentials, and $T_{X/Y} = Hom_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ the sheaf of relative tangent vector fields. We have the inclusion of the tangential branch locus in the branch locus

$$B^t_\pi = \{x \in X \mid T_{X/Y,x} \text{ is not free}\} \subset B_\pi = B_{X/Y} = \{x \in X \mid \Omega_{X/Y,x} \text{ is not free}\}.$$

Define as in [5, Definitions 17.1.1 and 17.3.1] a morphism π to be formally smooth at a point x in X if the induced map of local rings $\mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$ is formally smooth, and that π is smooth at x if it is locally finitely presented and formally smooth; say also that π is smooth if it is smooth at all points in X. In the light of the Jacobian criterion, namely that $B_\pi = \emptyset$, goes a long way to implying that the morphism π is smooth (*Thms. 3.1 and 3.3*), it is a natural to ask, with Zariski and Lipman [12], what are the implications of $B^t_\pi = \emptyset$? The example $X = \operatorname{Spec} A[x]/(x^2) \to Y = \operatorname{Spec} A$, i.e. the scheme of dual numbers over a commutative ring A, shows that if we want π to be smooth, the condition $B^t_\pi = \emptyset$ needs at least to be supplemented with the condition that the rank of $T_{X/Y}$ equals the relative dimension at each point in X, which can be imposed by

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assuming that X/Y is smooth at generic points in X. It is a remarkable fact that although $T_{X/Y}$ cannot even directly detect torsion in $\Omega_{X/Y}$, it turns out that these conditions combined imply $B_\pi = \emptyset$ (and hence imply that π is smooth) in interesting cases in characteristic 0. Already the result that $B^t_{V/k} = 0$ implies smoothness when V/k is a curve over a field of characteristic 0, due to Lipman [loc. cit], is, I think, quite surprising and non-trivial (see Proposition 4.4). In positive characteristic it is easy to see that smoothness at points of height ≤ 1 does not follow from $B^t_\pi = \emptyset$, so one could perhaps add the assumption $\operatorname{codim}_X B_\pi \geq 2$; but this is still not enough. What is needed is a condition on the discriminant locus $D_\pi = \pi(B_\pi)$. Before the main results are presented we describe some terminology.

Generalities: All schemes are assumed to be nœtherian and we use the notation in EGA, but see also [13, §5] and [8]. The height $\operatorname{ht}_X(x)$ of a point x in X is the same as the Krull dimension of the local ring $\mathcal{O}_{X,x}$ at x, and the dimension of X is defined as $\dim X = \sup\{\ \operatorname{ht}(x) \mid x \in X\}$. The dimension at a point x in X is

$$\dim_x X = \sup\{\ \operatorname{ht}(x_1) \mid x_1 \in X \text{ and } x \text{ specialises to } x_1\};$$

see [4, Proposition 5.1.4]. A point x in a subset T of X is *maximal* if for each point y in T that belongs to the closure $\{x\}^-$ of $\{x\}$ (in other words, x specialises to y (see [8, p. 93])), we have $\operatorname{ht}(x) \leq \operatorname{ht}(y)$. That is, if $x' \in T$ specialises to x, and $\operatorname{ht}(x') \leq \operatorname{ht}(x)$, then $x' = x$. Denote by $\operatorname{Max}(T)$ the set of maximal points of T, so $\operatorname{Max}(X)$ consists of points of height 0. A property on X is *generic* if it holds for all points in $\operatorname{Max}(X)$. Put

$$\begin{aligned} \operatorname{codim}^+_X T &= \sup\{\ \operatorname{ht}(x) \mid x \in \operatorname{Max}(T)\}, \\ \operatorname{codim}^-_X T &= \inf\{\ \operatorname{ht}(x) \mid x \in \operatorname{Max}(T)\}, \end{aligned}$$

so $\operatorname{codim}^-_X T \leq \operatorname{ht}(x) \leq \operatorname{codim}^+_X T$ when $x \in \operatorname{Max}(T)$. If T is the empty set, put $\operatorname{codim}^+_X T = -1$ and $\operatorname{codim}^-_X T = \infty$, since we are interested in lower and higher bounds on $\operatorname{codim}^\pm_X T$, respectively. For a coherent $\mathcal{O}_X$-module M, the stalk at a point x is denoted M_x and we put $\operatorname{depth}_T M = \inf\{\operatorname{depth} M_x \mid x \in T\}$. The fibre X_y over a point y in Y is the fibre product $\operatorname{Spec} k_{Y,y} \times_Y Y$, where $k_{Y,y}$ is the residue field at y. We define the *relative dimension* $d_{X/Y,x}$ of π at a point $x \in X$ as the infimum of the dimension of the vector space of Kähler differentials at all maximal points ξ that specialise to x, i.e.

$$d_{X/Y,x} = \inf\{\dim_{k_{X,\xi}} k_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \Omega_{X/Y,\xi} \mid x \in \{\xi\}^-, \xi \in \operatorname{Max}(X)\}.$$

To understand this number it is useful recall that

$$\begin{aligned} \dim_{k_{X,\xi}} k_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \Omega_{X/Y,\xi} &= \dim_{k_{X_{\pi(\xi)},\xi}} \Omega_{X_{\pi(\xi)}/k_{Y,\pi(\xi)}} \\ &= \dim_{k_{X,\xi}} k_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{Y,\pi(\xi)}}; \end{aligned}$$

see Proposition 2.1 for the first equality, but note that in general the numbers $d_{X/Y,x}$ and $\dim_x X_{\pi(x)}$ are not equal. On the other hand, if π is flat at x, then

$\dim_x X_{\pi(x)} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,\pi(x)}$, and if morover π is smooth at all points $\xi \in \mathrm{Max}(X)$ that specialise to x, then $d_{X/Y,x} = \dim_x X_{\pi(x)}$.

Recall also (this is an easy extension of [8, Chapter II, Lemma 8.9]):

($*$) a coherent $\mathcal{O}_X$-module M is free at a point x if M_ξ is free of rank equal to $\dim_{k_{X,x}} k_{X,x} \otimes_{\mathcal{O}_{X,x}} M_x$ for each $\xi \in \mathrm{Max}(X)$ that specialises to x.

Theorem 1.1. *Let $X \to S$ and $Y \to S$ be morphisms of nœtherian schemes which are of locally of finite type, and $\pi : X/S \to Y/S$ be a flat S-morphism. Assume that the branch loci $B_{X/S} = B_{Y/S} = \emptyset$ (e.g. X/S and Y/S are smooth), X is Cohen-Macaulay, and $\mathrm{codim}_Y^- D_\pi \geq 1$. If $B_\pi^t = \emptyset$, then for each point $y \in Y$*

$$\mathrm{codim}_{X_y}^+ B_{X_y/k_{Y,y}} \leq 1.$$

Remark 1.2. The condition of generic smoothness, i.e. $\mathrm{codim}_Y^- D_{X/Y} \geq 1$, is satisfied when $\mathcal{O}_{X,x}$ is regular and the extension of residue fields $k_{X,x}/k_{Y,\pi(x)}$ is separable for all points x such that $\pi(x) \in \mathrm{Max}(Y)$.

Corollary 1.3. *Let V/k be a variety defined by a regular sequence $\{f_1, \dots, f_r\}$ in some polynomial ring $k[X_1, \cdots, X_n]$ and assume that $T_{V/k}$ is locally free.*

(1) If $\mathrm{Char}\, k = 0$ then V/k is smooth.

(2) If $\mathrm{Char}\, k > 0$, assume moreover that the ring $k(f_1, \dots, f_r) \otimes_{k[f_1,\dots,f_r]} k[X_1, \dots, X_n]$ is smooth over the field $k(f_1, \dots, f_r)$. Then

$$\mathrm{codim}_V^+ B_{V/k} \leq 1.$$

In characteristic 0, Scheja and Storch [17] proved Corollary 1.3 when V/k is a hypersurface (we include a proof); Moen [14] proved it when V is a homogeneous complete intersection; Hochster [10] proved it when V is the spectrum of a graded ring, and he even attempted to find a counter-example when V/k is a locally complete intersection surface. Platte [16] found an elementary proof in the graded case, applicable also to analytic algebras. Using the existence of weakly submersive resolutions of singularities (work of Hironaka) combined with a detailed study of mixed Hodge structures, Straten and Steenbrink [19] argue that $\mathrm{codim}_X^+ B_{X/\mathbf{C}} \leq 2$ when $T_{X/\mathbf{C}}$ is locally free and X is an analytic space with at most isolated singularities; this was extended to non-isolated singularities by Flenner [2].

I want to thank the careful referee for helping me make this paper more accessible.

2 Base change for relative tangent vector fields

Let $\pi : X \to Y$ be a morphism of nœtherian schemes which is locally of finite type and generically smooth, so $\Omega_{X/Y,x}$ is free of rank $d_{X/Y,x}$ when x is a maximal point. The branch scheme $B^{(i)}_{X/Y}$, $i = 0, \dots,$ is defined by the Fitting

ideal $F_{d_{X/Y}+i}(\Omega_{X/Y})$, and $B_\pi = B_{X/Y} = V(F_{d_{X/Y}}(\Omega_{X/Y}))$. Similarly, the tangential branch scheme is defined by $B^t_{X/Y} = V(F_{d_{X/Y}}(T_{X/Y}))$ (see [6, Sec. 1.4, p. 21], [7, Ch. 20] and [11]). We will study base change diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{j} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ Y_1 & \longrightarrow & Y, \end{array} \tag{BC}$$

where $X_1 = X \times_Y Y_1$. I was unable to find a good reference for the following well-known important fact (see however [5, Proposition 16.4.5]).

Proposition 2.1. *Consider the diagram (BC).*

(1) The canonical morphism

$$j^*(\Omega_{X/Y}) \to \Omega_{X_1/Y_1}$$

is an isomorphism.

(2) $B^{(i)}_{X_1/Y_1} = B^{(i)}_{X/Y} \times_X X_1$.

(3) Consider the canonical morphism

$$\psi : j^*(T_{X/Y}) \to T_{X_1/Y_1}.$$

If ψ is an isomorphism, then

$$B^t_{X_1/Y_1} = B^t_{X/Y} \times_X X_1.$$

Proof. For the proof we can assume that all schemes are affine, so let $A \to B$ and $A \to A_1$ be homomorphisms of commutative rings, and put $B_1 = A_1 \otimes_A B$.

(1): Let $d_{B/A} : B \to \Omega_{B/A}$ be a universal derivation and define the A_1-linear derivation $d = \mathrm{id} \otimes d_{B/A} : B_1 \to A_1 \otimes_A \Omega_{B/A}$, which can be factorised over a universal derivation $d_{B_1/A_1} : B_1 \to \Omega_{B_1/A_1}$ by a B_1-homomorphism $\tilde{d} : \Omega_{B_1/A_1} \to A_1 \otimes_A \Omega_{B/A}$. There exists a natural B_1-homomorphism $p : A_1 \otimes_A \Omega_{B/A} \to \Omega_{B_1/A_1}$, which is the inverse of $\tilde{d}$.

(2): By (1), $\Omega_{B_1/A_1} = B_1 \otimes_B \Omega_{B/A}$. Let $F(\Omega_{B/A})$ denote the Fitting ideal defining $B^{(i)}_{B/A}$ and recall that $B_1F(M) = F(B_1 \otimes_B M)$ for a B-module M of finite type. Then

$$\begin{aligned} & B^{(i)}_{X/Y} \times_X X_1 = V(F(\Omega_{B/A})) \times_{\operatorname{Spec} B} \operatorname{Spec} B_1 = \operatorname{Spec}(B/F(\Omega_{B/A}) \otimes_B B_1) \\ & = \operatorname{Spec} \frac{B_1}{B_1F(\Omega_{B/A})} = \operatorname{Spec} \frac{B_1}{F(B_1 \otimes_B \Omega_{B/A})} = \operatorname{Spec} \frac{B_1}{F(\Omega_{B_1/A_1})} = B^{(i)}_{X_1/Y_1}. \end{aligned}$$

(3) is proven in the same way as (2). $\square$

If $Y_1 \to Y$ is flat or $B_{X/Y} = \emptyset$, then the canonical homomorphism ψ is an isomorphism [5, Proposition 16.5.11], but in general it need be neither injective nor surjective, contrary to the good behaviour of $\Omega_{X/Y}$.

Proposition 2.2. *Assume that* $\operatorname{codim}_X^- B_{X/Y} \geq 2$ *and* $\operatorname{codim}_{X_1}^- B_{X_1/Y_1} \geq 2$, *and that X and X_1 satisfy (S_2) at all points in $j(X_1)$. Then the canonical morphism ψ is an isomorphism.*

Of course, if π is flat, X_1 satisfies (S_2) and Y satisfies (S_2) along Y_1, then X satisfies (S_2) along $j(X_1)$.

Lemma 2.3. *(1) Let B be a ring, I an ideal, and N and M B-modules (not necessarily of finite type). If there exists an N-regular sequence in I of length 2, then* $\operatorname{depth}_I \ Hom_R(M,N) \geq 2$.

In particular, if $\operatorname{depth}_I R \geq 2$, *then* $\operatorname{depth}_I T_{R/k} \geq 2$ *for any subring $k \subset R$.*

(2) Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a flat homomorphism of local rings, where A is regular. Let N be a B-module of finite type which is flat over A. If $\operatorname{depth}_{\mathfrak{m}_B} N/\mathfrak{m}_A N \geq 1$ *and* $\operatorname{depth}_{\mathfrak{m}_B} N \geq 2$, *then* $\operatorname{depth}_{\mathfrak{m}_B} \mathfrak{m}_A Hom_B(M,N) \geq 2$.

Proof. (1): Let $\{x_1, x_2\}$ be an N-regular sequence in I. Clearly, x_1 is $Hom_B(M,N)$-regular. Assume $x_2\phi_2(m) = x_1\phi_1(m)$, $\phi_i \in Hom_B(M,N)$, $m \in M$. Since $\{x_1, x_2\}$ is N-regular, $\phi_2(m) = x_1 n'$, $n' \in N$. Since x_1 is a regular element this gives a well-defined homomorphism $\phi' \in Hom_B(M,N)$, $\phi'(m) = n'$, and $\phi_2 = x_1\phi'$, hence $\{x_1, x_2\}$ is $Hom_B(M,N)$-regular. Therefore

$$Ext^i_R(k_P, Hom_R(M,N)) = 0$$

when $i \leq 1$ and k_P is the residue field at a prime of height ≥ 2.

(2): Let x_1 be $N/\mathfrak{m}_A N$-regular and $\{x_1, x_2\}$ be an N-regular sequence. Since A is regular $\mathfrak{m}_A = (y_1, \ldots, y_r)$ where $\{y_1, \ldots, y_r\}$ is an A-regular sequence, and since N is flat it is also an N-regular sequence. Then $\{y_1, \ldots, y_r, x_1\}$ is an N-regular sequence. Assume $x_2\phi_2 = x_1\phi_2$, where $\phi_1, \phi_2 \in \mathfrak{m}_A Hom_B(M,N)$. As $\{x_1, x_2\}$ is N-regular, $\phi_2(m) \in x_1 N$, and since x_1 is N-regular $\phi_2 = x_1\phi_2'$, where $\phi_2' \in Hom_B(M,N)$. Therefore $\phi_2 \in x_1 Hom_B(M,N) \cap \mathfrak{m}_A Hom_B(M,N)$. Assume that $\sum y_i f_i = x_1 f$, $f, f_i \in Hom_B(M,N)$. Since $\{y_1, \ldots, y_r, x_1\}$ is N-regular we have $x_1 N \cap \mathfrak{m}_A N = x_1 \mathfrak{m}_A N$, hence $f_i(m) \in x_1 N$, and since x_1 is N-regular, $f_i = x_1 f_i'$ where $f_i' \in Hom_B(M,N)$. This implies $\phi_2 \in x_1 Hom_B(M,N) \cap \mathfrak{m}_A Hom_B(M,N) = x_1 \mathfrak{m}_A Hom_B(M,N)$, and thus $\{x_1, x_2\}$ is $\mathfrak{m}_A Hom_B(M,N)$-regular. $\square$

Proof of Proposition 2.2. Let $i : X^0 = X \setminus B_{X/Y} \to X$ be the inclusion morphism, $j_0 : X_1^0 \to X^0$ the base-change of j over i, and let $i_1 : X_1^0 \to X_1$ be the canonical morphism, so that $i \circ j_0 = j \circ i_1$. We have:

(i) Ω_{X_0/Y_0} is locally free so $j_0^*(T_{X^0/Y}) = T_{X_1^0/Y_1}$.

(ii) j is quasi-compact and separated, so cohomology commutes with base-change over the flat morphism i, in particular $j^* \circ i_* = (i_1)_* \circ j_0^*$ as a functor on quasi-coherent sheaves on X^0.

(iii) $\operatorname{codim}^-_X B_{X/Y} \geq 2$ and X satisfies (S_2), so $T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ satisfies (S_2) (*Lem.* 2.3), implying $T_{X/Y} = i_* i^*(T_{X/Y})$; similarly, since X_1 satisfies (S_2) and $\operatorname{codim}^-_{X_1} B_{X_1/Y_1} \geq 2$ we have $T_{X_1/Y_1} = (i_1)_* i_1^*(T_{X_1/Y_1}) = (i_1)_*(T_{X_1^0/Y_1})$.

(i-iii) imply

$$\begin{aligned} j^*(T_{X/Y}) &= j^* i_* i^*(T_{X/Y}) = (i_1)_* j_0^* i^*(T_{X/Y}) = (i_1)_* j_0^*(T_{X^0/Y}) = (i_1)_*(T_{X_1^0/Y_1}) \\ &= T_{X_1/Y_1}. \end{aligned}$$

□

Define the following subsets of X_1:

$$\mathcal{A} = \operatorname{supp} \operatorname{Ker}(\psi), \quad \mathcal{B} = \operatorname{supp} \operatorname{Coker}(\psi).$$

Proposition 2.4. *Let $\pi : X \to Y$ be a finitely presented morphism of schemes.*

(1) If π is flat, Y is regular, and X_1 contains no embedded associated components, then $\operatorname{codim}^+_{X_1} \mathcal{A} \leq 1$.

(2) If $\operatorname{Im}(\psi)$ satisfies (S_2) and X_1 contains no embedded associated component of codimension ≥ 2 (e.g. X_1 is normal), then $\operatorname{codim}^+_{X_1} \mathcal{B} \leq 1$.

(3) Assume that ψ is injective, X_1 satisfies (S_2), and $\operatorname{codim}^-_{X_1} B_{X_1/Y_1} \geq 2$. If $T_{X/Y}$ is locally free, then T_{X_1/Y_1} is locally free.

Proof. (1): Let $\bar{\psi} : j^{-1}(T_{X/Y}) \to T_{X_1/Y_1}$ be the natural morphism. We have $\mathfrak{m}_{Y,\pi(x)} j^{-1}(T_{X/Y,x}) \subset \operatorname{Ker}(\bar{\psi})_x$ for each point x. Assume on the contrary that $x \in \operatorname{Max}(\mathcal{A}) \subset X_1$ is a point of height ≥ 2, so $\mathfrak{m}_{Y,\pi(x')} T_{X/Y,x'} = \operatorname{Ker}(\psi)_{x'}$ when $\operatorname{ht}(x') \leq 1$ and x is a specialisation of x'. By Lemma 2.3 depth $\mathfrak{m}_{Y,\pi(x)} j^{-1}(T_{X/Y,x}) \geq 2$, hence $\mathfrak{m}_{Y,\pi(x)} j^{-1}(T_{X/Y,x}) = \operatorname{Ker}(\bar{\psi})_x$, implying ψ_x is injective, and contradicting the assumption $x \in \mathcal{A}$.

(2): Assume on the contrary that there exists a point $x \in \operatorname{Max}(\mathcal{B}) \subset X_1$ of height ≥ 2. Since $\operatorname{Im}(\psi)_x$ has depth ≥ 2 the exact sequence $0 \to \operatorname{Im}(\psi)_x \to T_{X_1/Y_1,x} \to \operatorname{Coker}(\psi)_x \to 0$ is split, so we get an injective homomorphism $\operatorname{Coker}(\psi)_x \to T_{X_1/Y_x,x}$, and since $x \in \operatorname{Max}(\mathcal{B})$, it follows that $\operatorname{Coker}(\psi)_x$ can be identified with a submodule of codimension ≥ 2 in $T_{X_1/Y_1,x}$. By assumption $\mathcal{O}_{X_1,x}$ contains no associated prime of height ≥ 2, hence $T_{X_1/Y_1,x} = \mathcal{H}om_{\mathcal{O}_{X_1,x}}(\Omega_{X_1/Y_1,x}, \mathcal{O}_{X_1,x})$ also has no associated prime of height ≥ 2, which gives a contradiction.

(3): Since $j^*(T_{X/Y})$ is locally free and X_1 satisfies (S_2), $j^*(T_{X/Y})$ satisfies (S_2), and the conditions in (2) are satisfied, so $\operatorname{codim}_{X_1} \mathcal{B} \leq 1$. Since $\operatorname{codim}_{X_1} B_{X_1/Y_1} \geq 2$ it follows that $\mathcal{B} = \emptyset$, hence ψ is an isomorphism. □

Proposition 2.5. *Let $\pi : X \to Y$ be a flat locally of finite type morphism of nœtherian schemes and consider the base change diagram (BC). Assume that X_1/Y_1 and X/Y are generically smooth, X_1 and Y satisfies (S_2), and* $\operatorname{codim}^-_{X_1} B_{X_1/Y_1} \geq 2$.

(1) *In a neighbourhood of $j(X_1)$, $\operatorname{codim}^-_X B_{X/Y} \geq 2$ and X satisfies (S_2).*

(2)
$$B^t_{X_1/Y_1} = B^t_{X/Y} \times_X X_1.$$
In particular, the module T_{X_1/Y_1} is locally free if and only if $T_{X/Y}$ is locally free in a neighbourhood of $j(X_1)$.

Proof. (1): It is well-known that by flatness, X satisfies (S_2) at points in $j(X_1) \subset X$. When $Y_1 \to Y$ is flat the assertion is obvious, so by Stein factorisation we can assume that $Y_1 \to Y$ is a closed immersion, hence $j : X_1 \to X$ is a closed immersion. The assertion is also obvious when $D_\pi \cap i(Y_1) = \emptyset$, so assume that there exists a point x in $B_{X/Y}$ that specialises to a point $x_0 \in j(X_1)$. We can assume that $x \in \operatorname{Max}(B_{X/Y})$ and we can also find $x_1 \in \operatorname{Max}(B_{X_1/Y_1})$, such that $j(x_1)$ specialises to x_0 and x specialises to $j(x_1)$, and $j(x_1) \in \operatorname{Max} j(B_{X_1/Y_1}) = \operatorname{Max}(j(X_1) \cap B_{X/Y})$. In other words, x_1 is a maximal point in the set of points in $j(X_1) \subset X$ that are specialisations of the point x, therefore by the going-down theorem for flat morphisms
$$\operatorname{ht}_{X_1}(x_1) \leq \operatorname{ht}_X(x).$$
If on the contrary $\operatorname{ht}_X(x) \leq 1$, then $x_1 \notin B_{X_1/Y_1}$ so $\Omega_{X_1/Y_1,x_1}$ is free of rank $d_{X_1/Y_1,x_1}$. Since X_1/Y_1 is generically smooth and locally of finite type, $d_{X_1/Y_1,x_1}$ equals the Krull dimension of a generic fibre of X_1/Y_1, and since X/Y is flat the Krull dimension of the generic fibres of X/Y that specialise to the same generic fibre of X_1/Y_1 also equals $d_{X_1/Y_1,x_1}$, and as X/Y is generically smooth, we conclude that $d_{X/Y,j(x_1)} = d_{X_1/Y_1,x}$. Now since $\Omega_{X_1/Y_1,x_1} = j^*(\Omega_{X/Y})_{x_1}$ (*Prop.* 2.1, (1)), it follows that the $\mathcal{O}_{X,j(x)}$-module $\Omega_{X/Y,j(x_1)}$ is generated by $d_{X/Y,j(x_1)}$ elements, implying that $\Omega_{X/Y,j(x_1)}$ is free, and hence $j(x_1) \notin B_{X/Y}$ (*Prop.* 2.1, (2)). Since x specialises to $j(x_1)$ it follows that $x \notin B_{X/Y}$, resulting in a contradiction. Therefore $\operatorname{ht}_X(x) \geq 2$.

(2): By (1) and Proposition 2.2 the canonical morphism $\psi : j^*(T_{X/Y}) \to T_{X_1/Y_1}$ is an isomorphism so the assertion is implied by Proposition 2.1. □

3 Differential criterion of smoothness

The relation between the branch locus and the locus of non-smooth points of a morphism is of course much discussed in the literature, but there still seems to remain room for clarification. In [18, §2] one can find a nice summary of characterisations of smoothness in terms of the vanishing of André-Quillen homology and the Jacobian condition $B_\pi = \emptyset$. We are however more interested in the "Jacobian characterisation" [5, Proposition 17.15.15]. A proof is included

since I find the argument in EGA difficult to disentangle and I could not find any other satisfactory treatment of this important result in the literature. The proof relies on the fundamental theorem that smoothness implies flatness, and conversely, if a morphism is flat at a point and smooth along the fibre at the point, then the morphism is smooth at that point [3, Théorème 19.7.1].

Theorem 3.1. *Let $\pi : X \to Y$ be a morphism of schemes which is locally finitely presented. Let x be a point in X and put $y = \pi(x)$. The following are equivalent:*

(1) π is smooth at the point x.

(2) π is flat at x, $x \notin B_\pi$, and is smooth at all points $\xi \in \mathrm{Max}(X)$ that specialises to x.

(3) π is flat at x, $x \notin B_\pi$, and $\mathrm{rk}\,\Omega_{X/Y,x} = \dim_x X_y$.

(4) π is flat at x, $x \notin B_\pi$, and $X_y/k_{Y,y}$ is smooth at all points $\eta \in \mathrm{Max}(X_y)$ that specialises to x.

Remark 3.2. The proof in [5] that the rank of $\Omega_{X/Y,x}$ is as asserted in (3) seems to contain a gap. It relies on [5, Proposition 17.10.2], and presupposes that π be smooth not only at the point x, but also at all specialisations of x.

Proof. (1) $\Rightarrow$ (2) and (3): Put $A = \mathcal{O}_{Y,y}$ and $B = \mathcal{O}_{X,x}$, so $A \to B$ is formally smooth, and since moreover this homomorphism is finitely presesented, it is also flat [3, Th 17.5.1]. Let ξ be a point that specialises to x, and put $B' = \mathcal{O}_{X,\xi}$ and $A' = \mathcal{O}_{Y,\pi(\xi)}$. It follows directly from the definition of formal smoothness that the composition $A \to B \to B'$ and the base change $A' \to B'$ are also formally smooth (see also [13, Theorem 28.2]); hence π is smooth at ξ. Since $A \to B$ is formally smooth it follows that $\Omega_{B/A}$ is projective [13, Theorem 28.5], and since B is a local ring, $\Omega_{B/A}$ is free; hence $x \notin B_\pi$. We now determine the rank $r = \mathrm{rank}\,\Omega_{B/A}$. Let k be an algebraic closure of $k_{Y,y}$, put $\bar{B} = k \otimes_{k_{Y,y}} k_{Y,y} \otimes_A B$, and let k_B be the residue field of $\bar{B}$. Proposition 2.1 implies that $\Omega_{\bar{B}/k}$ is free of rank r, hence by the second fundamental exact sequence in [13, Theorem 25.2], noting that k_B/k is formally smooth since k is algebraically closed,

$$r = \dim_k \frac{\mathfrak{m}_{\bar{B}}}{\mathfrak{m}_{\bar{B}}^2} + \mathrm{tr.deg}\, k_B/k.$$

Formal smoothness is preserved under base change, hence the map $k \to \bar{B}$ is formally smooth; hence $\bar{B}$ is a regular local ring [13, Lemma 1], so $\dim_k \frac{\mathfrak{m}_{\bar{B}}}{\mathfrak{m}_{\bar{B}}^2} = \dim \bar{B}$. Let now x_1 be a point in X_y such that $\dim_x X_y = \mathrm{ht}_{X_y}(x_1) = \dim \mathcal{O}_{X_y,x_1}$. Put $R = k \otimes_{k_{Y,y}} \mathcal{O}_{X_y,x_1}$ and let k_R denote its residue field. Then $\bar{B} = R_P$ for some prime ideal P in R, and $\dim_x X_y = \dim R$, since R is a flat base change of $k_{Y,y} \to \mathcal{O}_{X_y,x_1}$. Since, moreover, π is finitely presented and k is algebraically closed, it follows that $k = k_R$, and by Hilbert's Nullstellensatz [13, Theorem 5.6] $\dim R/P = \mathrm{tr.deg}\, k_B/k$. Since R is a catenary ring, we then get

$$\dim R = \mathrm{ht}_{X_y}(x) + \dim R/P = \dim \bar{B} + \mathrm{tr.deg}\, k_B/k = r.$$

(2) $\Rightarrow$ (1): Since π is flat at x, it suffices by [3, Th 19.7.1] to prove that $k_{Y,y} \to k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X_y,x}$ is formally smooth to conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is formally smooth, and by [13, Theorem 30.3] (see also [5, Proposition 15.15.5]) this follows if $\Omega_{\mathcal{O}_{X_y,x}/k_{Y,y}}$ is free of rank $\dim_x X_y$.

Since $x \notin B_\pi$, by Proposition 2.1 the $\mathcal{O}_{X_y,x}$-module $\Omega_{X_y/k_{Y,y},x}$ is free and $\operatorname{rk}\Omega_{X_y/k_{Y,y},x} = \operatorname{rk}\Omega_{X/Y,x}$. Since π is smooth at points $\xi \in \operatorname{Max}(X)$ that specialise to x, and $\Omega_{X/Y,\xi} = \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,x}} \Omega_{X/Y,x}$, it follows as in the proof of (1) $\Rightarrow$ (3) that this rank equals $\dim_\xi X_{\pi(\xi)}$ (or see [5, Prop. 17.15.5]). Since π is flat, $\dim_x X_y = \dim_\xi(X_{\pi(\xi)})$, hence $\operatorname{rk}\Omega_{X_y/k_{Y,y},x} = \dim_x X_y$.

(3) $\Rightarrow$ (4): Since $\Omega_{X/Y,x}$ is free of rank $\dim_x X_y$ it follows that $\Omega_{X_y/k_{Y,y},x}$ is free of rank $\dim_x X_y$ (*Prop.* 2.1). In the same way as in the proof of (2) $\Rightarrow$ (1) it follows that $k_{Y,y} \to \mathcal{O}_{X_y,x}$ is formally smooth; hence any localisation $k_{Y,y} \to \mathcal{O}_{X_y,\eta}$ is also formally smooth.

(4) $\Rightarrow$ (1): Since $k_{Y,y} \to \mathcal{O}_{X_y,\eta}$ is formally smooth it follows that $\Omega_{X_y/k_{Y,y},\eta}$ is free of rank $\dim_\eta X_y = \dim_x X_y$. Now $\Omega_{X/Y,x}$ is free and by Proposition 2.1 its rank is $\dim_x X_y$. It follows as in the proof of (2) $\Rightarrow$ (1) that the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is formally smooth. $\square$

Often the condition (2) or (3) in Theorem 3.1 serve as a *definition* of smoothness (see e.g. [8]). Alternatively, π is smooth if it is flat and all its fibres are smooth [5, Théorème 17.5.1]. In either case, the condition that π be flat can be a nuisance. Put $\Gamma_{X/Y/S} = \operatorname{Ker}(\pi^*(\Omega_{Y/S}) \to \Omega_{X/S})$. Assuming X/S is smooth, (3) in the following theorem shows that the non-smoothness locus of π is exactly $\operatorname{supp}\Gamma_{X/Y/S} \cup B_\pi$. Therefore, if $\Gamma_{X/Y/S} = 0$ it follows that the Jacobian criterion $B_\pi = \emptyset$ implies smoothness, i.e. flatness is automatic. Moreover, $\Gamma_{X/Y/S} = 0$ when either X/Y is generically smooth and a locally complete intersection, or X/S is a locally complete intersection (see [11, Proposition 2.11] for a discussion of this assertion).

Theorem 3.3. *Assume that X/S is smooth at the point x. The following are equivalent:*

(1) $\pi : X \to Y$ is smooth at x.

(2) $\pi^*(\Omega_{Y/S})_x \xrightarrow{p} \Omega_{X/S,x}$ has a left inverse.

(3) $x \notin B_\pi$ and $\Gamma_{X/Y/S,x} = 0$.

Proof. (1) $\Leftrightarrow$ (2): See [5, Th. 17.11.1]. (3) $\Rightarrow$ (2): This can be seen directly from the fundamental exact sequence of differentials [8, Proposition 8.11]. (1) $\Rightarrow$ (3): (1) implies by Theorem 3.1 that $x \notin B_\pi$, and since (1) implies (2) we also get $\Gamma_{X/Y/S,x} = 0$. $\square$

4 Proof of main results

Although our main result does not rely on the following preliminary result, it does provide insight into how the assumption 'locally complete intersection' is applied.

Lemma 4.1. *(Lichtenbaum-Schlessinger* [12, Prop. 5.2]*) Let X/k be a l.c.i. scheme locally of finite type over a field. Assume that X/k is generically smooth (e.g. k is perfect and X is reduced) and $T_{X/k}$ is locally free. Then $\operatorname{codim}^+_X B_{X/k} \leq 2$.*

Proof. We follow the proof in [loc. cit.]. The problem being local at a point x, we can assume that there exists a regular immersion $i : X \to X_r$ over k, where X_r/k is smooth. Since I/I^2 is locally free and X/k is generically smooth we have the presentation $0 \to I/I^2 \to i^*(\Omega_{X_r/k}) \to \Omega_{X/k} \to 0$, so $\operatorname{p.d.}\Omega_{X/k,x} \leq 1$. So $\Omega_{X/,k}$ is free if and only if $Ext^1_{\mathcal{O}_{X,x}}(\Omega_{X/k,x}, \mathcal{O}_x) = Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)_x = 0$, and therefore $B_{X/k} = \operatorname{supp} Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$. Assume now that $x \in \operatorname{Max}(B_{X/k})$, so x is an associated point of $Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ and therefore $\operatorname{depth} Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)_x = 0$. Dualising the presentation gives the exact sequence $0 \to T_{X/k} \to i^*(\Omega_{X_r/k})^* \to (I/I^2)^* \to Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \to 0$, since $\Omega_{X_r/k}$ is locally free; hence

$$\operatorname{p.d.} Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)_x \leq 2.$$

By the Auslander-Buchsbaum formula, $\operatorname{depth}\mathcal{O}_{X,x} = \operatorname{depth} Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)_x + \operatorname{p.d.} Ext^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)_x \leq 2$, and since X is Cohen-Macaulay, $\operatorname{ht}_X(x) \leq 2$. □

There is also a relative version for morphisms between smooth schemes, which is used in the proof of the main theorem.

Proposition 4.2. *Let X be a Cohen-Macaulay scheme and X/S and Y/S be smooth S-schemes. If $B^t_{X/Y} = \emptyset$, then*

$$\operatorname{codim}^+_X B_\pi \leq 2.$$

Lemma 4.3. *Assume that X/S and Y/S are smooth morphisms and that X/Y is generically smooth. Then $B_\pi = \operatorname{supp}\mathcal{C}_{X/Y}$.*

Proof. This follows from [11, Prop. 1.3], but we give a more concrete independent proof. Let $\phi : G_1 \to G_2$ be a homorphism of free modules over a local commutative ring A of rank g_1 and g_2, respectively. Put $M = \operatorname{Coker}(\phi)$ and $D(M) = \operatorname{Coker}(\phi^*)$ (the transpose of M). We assert:

$$M \neq 0 \Leftrightarrow \operatorname{Coker}(D(M)) \text{ cannot be generated by } g_1 - g_2 \text{ elements.} \qquad (**)$$

Letting $\bar{G}_1$ be the image of the homomorphism ϕ we get the exact sequence

$$0 \to \bar{G}_1^* \to G_1^* \to D(M) \to 0.$$

For an A-module N we denote by $\beta_i(N) = \dim_k Tor^i_A(k, N)$ the ith Betti number. The exact sequence results in a long exact sequence in homology, from which follows that

$$g_1 - \beta_0(D(M)) - \beta_0(\bar{G}_1^*) + \beta_1(D(M)) = 0,$$

so

$$\beta_0(D(M)) = g_1 - \beta_0(\bar{G}_1^*) + \beta_1(D(M)) \geq g_1 - \beta_0(\bar{G}_1^*) \geq g_1 - g_2,$$

where the last inequality is strict if and only if $\bar{G}_1 \neq G_2$.

To prove the Lemma, put $G_1 = T_{X/S,x}$, $G_2 = \pi^*(T_{Y/S})_x$ and $M = \mathcal{C}_{X/Y,x}$. Since X/S and Y/S are smooth it follows that $g_1 = d_{X/S,x}$, $g_2 = d_{Y/S,\pi(x)}$, and that $\Omega_{X/S,x}$ and $\Omega_{Y/S,\pi(x)}$ are free, hence $D(M) = \Omega_{X/Y,x}$. Since X/Y is generically smooth, $x \in B_\pi$ if and only if $\Omega_{X/Y,x}$ can be generated by $d_{X/Y,x}$ elements, and $d_{X/Y,x} = d_{X/S,x} - d_{Y/S,\pi(x)}$. By $(**)$, $\Omega_{X/Y,x}$ can be generated by $g_1 - g_2 = d_{X/S,x} - d_{Y/S,\pi(x)} = d_{X/Y,x}$ elements if and only of $\mathcal{C}_{X/Y,x} = 0$. $\square$

Proof of Proposition 4.2. Dualising the fundamental exact sequence of Kähler differentials [8, Proposition 8.11] results in the exact sequence

$$0 \to T_{X/Y} \to T_{X/S} \xrightarrow{d\pi} T_{X/S\to Y/S} \to \mathcal{C}_{X/Y} \to 0,$$

where $T_{X/S\to Y/S} = \mathit{Hom}_{\mathcal{O}_X}(\pi^*(\Omega_{Y/S}), \mathcal{O}_X)$, defining the normal module $\mathcal{C}_{X/Y}$. This is a locally free resolution of $\mathcal{C}_{X/Y}$, so $\mathrm{p.d.}\, \mathcal{C}_{X/Y,x} \leq 2$ at each point x in X. The local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay, therefore $\mathrm{ht}(x) = \mathrm{p.d.}\, \mathcal{C}_{X/Y,x}$ when $x \in \mathrm{Max}(\mathrm{supp}\, \mathcal{C}_{X/Y})$. Finally, since X/S and Y/S are smooth, $B_\pi = \mathrm{supp}\, \mathcal{C}_{X/Y}$ (*Prop.* 4.2), which completes the proof. $\square$

Proposition 4.4. *(Lipman* [12, Th. 1]*) Let X/k be a scheme locally of finite type over a field of characteristic* 0 *such that $B^t_{X/k} = \emptyset$. Then X is normal and in particular* $\mathrm{codim}^-_X B_{X/k} \geq 2$.

We include a sketch of the proof, following [loc cit], to clarify the situation in our notation.

Proof. One first proves that the module $C_X := \mathrm{Coker}(\Omega_{X/k} \xrightarrow{g} T^*_{X/k})$, where g is the biduality morphism, satisfies $\mathrm{depth}\, C_{X,x} \geq 2$ when $x \in \mathrm{supp}\, C_X$. Consider the exact sequence $0 \to \overline{\Omega}_{X/k} \to T^*_{X/k} \to C_X \to 0$, where $\overline{\Omega}_{X/k} = \mathrm{Im}(\Omega_{X/k} \to T^*_{X/k})$. Noting that $T_{X/k} = \Omega^*_{X/k} = \overline{\Omega}^*_{X/k}$, dualisation results in the exact sequence

$$0 \to C_X^* \to T^{**}_{X/k} \to T_{X/k} \to \mathit{Ext}^1_{\mathcal{O}_X}(C_X, \mathcal{O}_X) \to 0$$

since $T^*_{X/k}$ is locally free. As $T_{X/k}$ is reflexive we get

$$\mathit{Ext}^0_{\mathcal{O}_X}(C_X, \mathcal{O}_X) = C_X^* = 0 \quad \text{and} \quad \mathit{Ext}^1_{\mathcal{O}_X}(C_X, \mathcal{O}_X) = 0,$$

implying the assertion.

We always have $\mathrm{supp}\, C_X \subset B_{X/k}$. If $x \notin \mathrm{supp}\, C_X$, so the map $\Omega_{X/k,x} \to T^*_{X/k,x}$ is surjective, by a result of Nagata [15] there exist $\partial_i \in T_{X/k,x}$ and $x_j \in \mathfrak{m}_{X,x}$ such that $\partial_i(x_j)$ forms an invertible $d \times d$ matrix, where $d = \mathrm{ht}(x)$. Since $\mathrm{Char}\, k = 0$ it follows from the Zariski-Lipman-Nagata criterion that $\mathcal{O}_{X,x}$ is a regular ring, hence again since $\mathrm{Char}\, k = 0$, $x \notin B_{X/k}$. Therefore $\mathrm{supp}\, C_X = B_{X/k}$. By the first assertion in the Proposition, it follows that $\mathrm{depth}\, \mathcal{O}_{X,x} \geq 2$ when $x \in B_{X/k}$. Since regularity implies normality, so the locus of points where

X fails to be normal is contained in $B_{X/k}$, it follows that X is normal (either look at Lipman's nice argument in [loc cit, Prop 2.1] or think of Serre's normality criterion). $\square$

Proof of the assertion in Remark 1.2. Since X/Y is dominant and X is regular at all points in the generic fibres, and the problem is local at such fibres, it follows that we can assume that X and Y are integral. We will prove that if $\eta \in \mathrm{Max}(Y)$, then $\eta \notin D_{X/Y}$. Since X/Y is dominant, there exists $\xi \in \mathrm{Max}(X)$ such that $\pi(x) = \pi(\xi) = \eta$, and we can moreover let ξ be any maximal point that specialises to x, since it will satisfy $\pi(\xi) = \eta$ because η is maximal. We then have (as detailed below)

$$\begin{aligned}
\dim_{k_{X,x}} k_{X,x} \otimes_{\mathcal{O}_{X,x}} \Omega_{X/Y,x} &= \dim_{k_{X,x}} \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 + \dim_{k_{X,x}} \Omega_{k_{X,x}/k_{Y,\eta}} \\
&= \mathrm{ht}_X(x) + \mathrm{tr.deg}\, k_{X,x}/k_{Y,\eta} \\
&= \mathrm{ht}_Y(\eta) + \mathrm{tr.deg}\, k_{X,\xi}/k_{Y,\eta} \\
&= \mathrm{tr.deg}\, k_{X,\xi}/k_{Y,\eta} = \dim_{k_{X,\xi}} \Omega_{X/Y,\xi} = \mathrm{rank}\, \Omega_{X/Y,\xi}.
\end{aligned}$$

The first line follows since $k_{X,x}/k_{Y,\eta}$ is separable, hence 0-smooth, after applying the second fundamental exact sequence in [13, Theorem 25.2]. The second line follows since $\mathcal{O}_{X,x}$ is regular and since $k_{X,x}/k_{Y,\eta}$ is separable and finitely generated, so that a differential basis is the same as a transcendence basis (see [13, §26]). The third line follows since X and Y are integral, X/Y is locally of finite type integral, and Y is Nœtherian, so Ratliff's dimension equality holds [13, Theorem 15.6].

The see the second to last equality it suffices as above to prove that $k_{X,\xi}/k_{Y,\eta}$ is finitely generated and separable, where the finite generation follows since π is finitely presented at x. First we note that $\mathcal{O}_{X,\xi} = k_{X,\xi}$ since it is a localisation of the regular ring $\mathcal{O}_{X,x}$ and $\xi \in \mathrm{Max}(X)$. Secondly, $k_{X,x}/k_{Y,\eta}$ is separable and $\mathcal{O}_{X,x}$ is regular, hence $\mathcal{O}_{X,x}/k_{Y,\eta}$ is $\mathfrak{m}_{X,x}$-smooth [13, Lemma 1]. Therefore the localisation $\mathcal{O}_{X,\xi}/k_{Y,\eta}$ is $\mathfrak{m}_{X,\xi}$-smooth, hence $k_{X,\xi}$ is formally smooth over $k_{Y,\eta}$, implying that $k_{X,\xi}/k_{Y,\eta}$ is separable [13, Theorem 26.9].

The last equality follows since $\mathcal{O}_{X,\xi}$ is regular, so $\mathcal{O}_{X,\xi} = k_{X,\xi}$. Since the first and the last entries are equal it follows that $\Omega_{X/Y,x}$ is free (see $(*)$), so $\eta = \pi(x) \notin D_{X/Y}$. $\square$

Proof of Theorem 1.1. First assume that $x \in \mathrm{Max}(B_{X/Y})$ is such that $y = \pi(x) \in \mathrm{Max}(D_{X/Y})$. Thus, as $\mathrm{codim}_Y^- D_{X/Y} \geq 1$, $\mathrm{ht}_Y(y) \geq 1$, and by Proposition 4.2 $\mathrm{ht}_X(x) \leq 2$. We identify the fibre X_y with a subscheme of X. Select $x_1 \in \mathrm{Max}(X_y)$ that specialises to x. We have

$$\mathrm{ht}_{X_y}(x) = \mathrm{ht}_X(x) - \mathrm{ht}_X(x_1) = \mathrm{ht}_X(x) - \mathrm{ht}_Y(y) \leq 2 - 1 = 1,$$

where the first equality follows since X is catenary, and the second from flatness. This implies that $\mathrm{codim}_{X_y}^+ B_{X_y/k_{Y,y}} \leq 1$. Let now y_1 be an arbitrary point in $D_{X/Y}$. If $x_1 \in \mathrm{Max}(B_{X_{y_1}/k_{Y,y_1}})$ there exists a point $x \in \mathrm{Max}(B_{X/Y})$ that specialises to x_1, so $y = \pi(x) \in \mathrm{Max}(D_{X/Y})$. By the going-down theorem for

flat morphisms it follows that

$$\mathrm{ht}_{X_{y_1}}(x_1) \leq \mathrm{ht}_{X_y}(x),$$

and therefore $\operatorname{codim}^+_{X_{y_1}} B_{X_{y_1}/k_{Y,y_1}} \leq 1$. $\square$

Proof of Corollary 1.3. Put $A = k[y_1, \dots, y_r]$ and $B = k[X_1, \dots, X_n]$. If $\{f_1, \dots, f_r\} \subset k[X_1, \dots, X_n]$ is a regular sequence defining V, then V is a fibre of the flat morphism $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$, $y_i \mapsto f_i$ (see [13, Exercise 22.2]). If $x \in \operatorname{Max}(B_{V/k})$ and $\mathrm{ht}(x) \geq 2$ it follows from Proposition 2.5 that $T_{X/Y,x}$ is free, hence by Theorem 1.1 $\operatorname{codim}^+_V B_{V/k} \leq 1$. If moreover $\operatorname{Char} k = 0$, Proposition 4.4 implies that $B_{V/k} = \emptyset$. $\square$

Remarks 4.5. (1) The assumption in Corollary 1.3 that V/k is defined by a regular sequence is used to infer that the morphism $A \to B$ in the proof is flat (the local flatness criterion). Conversely, if $A \to B$ is flat then the fibre V is a complete intersection [9] (see [11] for a more general assertion).

(2) Zariski and Lipman [12] stated their conjecture only for varieties over fields of characteristic 0. We can "explain" the positive characteristic counterexample in [loc. cit, §7,b)]. The surface $V = V(XY - Z^p) \subset \mathbf{A}^3_k$ over a perfect field k of characteristic $p > 0$ is normal and $T_{V/k}$ is locally free. By normality and since k is perfect V is smooth at all points of height ≤ 1, in accordance with Corollary 1.3. Since V/k is not smooth at the origin, Theorem 1.1 implies that if V is the fibre of a flat family of surfaces $X \to Y$, where X/k and Y/k are smooth, then X/Y cannot be generically smooth in Y. For example, the hypersurface $X = V(t - XY - Z^p) \subset \mathbf{A}^4_k$ is smooth over k, the morphism $\pi : X \to Y = \mathbf{A}^1_k$ induced by the projection to the t-coordinate is flat, and $T_{X/Y}$ is locally free. However, π is not generically smooth since the field extension $k_{X,x}/k_{Y,\pi(x)}$ is not separable when x is the maximal point in X.

Hypersurfaces

Note that the proof of Corollary 1.3 is not obtained by reducing to hypersurfaces, and unlike Scheja and Storch's proof for hypersurfaces X/k over fields k of characteristic 0, we need not apply the Eagon-Northcott bound on heights of determinantal ideals. Since the proofs are that different, the proof of Corollary 1.3 can be compared to a more geometric version of their proof for hypersurfaces.

By Proposition 4.4 it suffices to prove $\operatorname{codim}^+_X B_{X/k} \leq 1$. Let $j : X/k \to Z/k$ be a regular immersion into a smooth variety Z/k, so locally $j(X)$ is defined by an ideal I such that I/I^2 is locally free over $\mathcal{O}_X$ and we have the short exact sequence $0 \to I/I^2 \to j^*(\Omega_{Z/k}) \to \Omega_{X/k} \to 0$, and in particular $\mathrm{p.d.}\, \Omega_{X/k,x} \leq 1$ for each point x in X. Dualising we get the exact sequences

$$0 \to T_{X/k} \to j^*(T_{Z/k}) \to \mathcal{C}_{X/Z} \to 0, \tag{E}$$

$$0 \to \mathcal{C}_{X/Z} \to (I/I^2)^* \to \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) \to 0. \tag{F}$$

We assert that that the normal module $\mathcal{C}_{X/Z}$ satisfies depth $\mathcal{C}_{X/Z,x} \geq 2$ when I is a locally principal ideal, $\mathrm{ht}_X(x) \geq 2$, and $x \in \mathrm{Max}(B_{X/k})$. A locally defined surjection $\mathcal{O}_Z^{d_X} \to j^*(T_{X/k})$ together with the surjective map $T_{Z/k} \to j_*j^*(T_Z/k)$ gives a lift

$$0 \to \mathcal{O}_Z^{d_X} \to T_{Z/k} \to \hat{\mathcal{C}}_{X/Z} \to 0 \tag{E'}$$

of the sequence (E), which is exact to the left since $T_{X/k}$ is locally free, and the cokernel $\hat{\mathcal{C}}_{X/Z} \subset \mathcal{O}_Z$ is an ideal. Since the ideal $I_{j(x)} = (f) \subset \mathcal{O}_{Z,j(x)}$ is locally principal, $\hat{\mathcal{C}}_{X/Z,j(x)} = T_{Z/k,j(x)} \cdot f \subset \mathcal{O}_{Z,j(x)}$. We note also that the element f belongs to the integral closure of the ideal $T_{Z/k,j(x)} \cdot f$; this is clear when $\mathrm{ht}_Z(j(x)) \leq 1$ and the general case follows from describing the integral closure of an ideal $J_z \subset \mathcal{O}_{Z,z}$ as the intersection $\cap J_z\mathcal{O}_{Z,z'}$, running over points z' such that $\mathrm{ht}_Z(z') \leq 1$ and z' specialises to z. Since $\mathrm{p.d.}\, \hat{\mathcal{C}}_{Z/X,j(x)} \leq 1$, by the Hilbert-Burch theorem $\hat{\mathcal{C}}_{X/Z,j(x)} = aF_1(\hat{\mathcal{C}}_{X/Z,j(x)})$ for some $a \in \mathcal{O}_{Z,j(x)}$; since $x \in \mathrm{Max}(B_{X/k})$ and $\mathrm{ht}_X(x) \geq 2$, the height of the ideal $\hat{\mathcal{C}}_{X/Z,j(x)} \subset \mathcal{O}_{Z,j(x)}$ is ≥ 2; hence by Krull's principal ideal theorem a is a unit; therefore $T_{Z/k,j(x)} \cdot f = \hat{\mathcal{C}}_{Z/X,j(x)} = F_1(\hat{\mathcal{C}}_{X/Z,j(x)})$. (Note: The weaker assertion $V_Z(T_{Z/k,j(x)} \cdot f) = V(F_1(\hat{\mathcal{C}}_{X/Z,j(x)}))$ is actually sufficient for the proof, and is easy to see: the germ $V(\hat{\mathcal{C}}_{X/Z,j(x)})$ is of codimension ≥ 2 and therefore $\hat{\mathcal{C}}_{X/Z,z}$ is not principal if and only if $z \in V(\hat{\mathcal{C}}_{X/Z,j(x)})$, by Krull's principal ideal theorem.) Considering germs of varieties at x we now get

$$\begin{aligned} V_X(F_1(\mathcal{C}_{X/Z})) &= V_Z(F_1(\hat{\mathcal{C}}_{X/Z}) + I) = V_Z(T_{Z/k} \cdot f + I) \\ &= V_Z(T_{Z/k} \cdot f) = V_Z(F_1(\hat{\mathcal{C}}_{X/Z})), \end{aligned}$$

so in particular $\mathrm{Max}(V_X(F_1(\mathcal{C}_{X/Z}))) = \mathrm{Max}(V_Z(F_1(\hat{\mathcal{C}}_{X/Z})))$, identifying X with $j(X)$. By the Eagon-Northcott bound on heights of determinant ideals [1] applied to the exact sequence (E'), we get $\mathrm{ht}_Z(z) \leq 2$ when $z \in \mathrm{Max}(V_Z(F_1(\hat{\mathcal{C}}_{Z/X})))$ and z specialises to $j(x)$. Therefore, if $j(x) \in \mathrm{Max}(V_X(F_1(\mathcal{C}_{X/Z})))$, we get $\mathrm{ht}_X(x) = \mathrm{ht}_Z(j(x)) - 1 \leq 1$. Since $\mathrm{ht}_X(x) \geq 2$ it follows that $\mathcal{C}_{X/Z,x}$ is free, and since X is Cohen-Macaulay, depth $\mathcal{C}_{X/Z,x} \geq 2$.

Assume now on the contrary that there exists a point $x \in \mathrm{Max}(B_{X/k})$ such that $\mathrm{ht}(x) \geq 2$. By the above result

$$Ext^1_{\mathcal{O}_X,x}(Ext^1_{\mathcal{O}_{X,x}}(\Omega_{X/k,x}, \mathcal{O}_{X,x}), \mathcal{C}_{X/Z,x}) = 0,$$

hence the sequence (F) splits, so there exists an injection $Ext^1_{\mathcal{O}_{X,x}}(\Omega_{X/k,x}, \mathcal{O}_{X,x}) \to (I_x/I_x^2)^*$. Since X is Cohen-Macauly, the free module $(I_x/I_x^2)^*$ has no embedded associated prime. Therefore

$$Ext^1_{\mathcal{O}_{X,x}}(\Omega_{X/k,x}, \mathcal{O}_{X,x}) = 0.$$

Since $\mathrm{p.d.}\, \Omega_{X/k,x} \leq 1$, this implies that $\Omega_{X/k,x}$ is free, contradicting the assumption that $x \in B_{X/k}$. Therefore $\mathrm{codim}^+_X B_{X/k} \leq 1$.

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